Descriptive Set Theory
Lecture 23

An example of use of the Effros space. Recall that every Polish space is a G_δ subset of $[0,1]^{\mathbb{N}}$, but more importantly, it is a closed subset of $\mathbb{R}^{\mathbb{N}}$. Thus, the Effros space $\mathcal{F}(\mathbb{R}^{\mathbb{N}})$ can be thought of as the space of all Polish spaces. In particular, one can study equivalence relations between Polish spaces (e.g. homeomorphism) using descriptive set theory because such equiv. rel. are often analytic subsets of $\mathcal{F}(\mathbb{R}^{\mathbb{N}})^2$.

Another example: For any Polish gp G, one can construct using $\mathcal{F}(G)$ a universal G-space, i.e. a Polish space with a continuous action of G on X s.t. any other Polish G-space is embedded in $G \curvearrowright X$ in a certain sense.

Determinacy of open/closed ... Borel games.

Determinacy of open/closed games (Gale-Stewart).

Open games and closed games are determined. I.e.

given an open/closed payoff set $P \subseteq A^{\mathbb{N}}$, for any $A \neq \emptyset$, one of the players has a winning strategy in $G(P)$.

Player 1: a_0 a_2 ... Player 1 wins if $(a_n) \in P$.
Player 2: a_1 a_3

Proof. For P open or closed, we call the player whose payoff set is the open set (P or P^c) **Open Player**, and we call the other player **Closed Player**.

Suppose the Open Pl. doesn't have a winning strategy and we show that the Closed Pl. does. We let Closed Pl. play a move where the Open Pl. still doesn't have a winning strategy: in the beginning Open Pl. doesn't have a winning strategy. Suppose that for the play $s \in A^{<\mathbb{N}}$ there is no winning strategy for the Open Pl. at s. If it's the Open Pl. turn, then $\nexists$ a move $a \in A$ s.t. sa is winning for the Open Pl. (i.e. $\exists$ winning strategy at sa), o.w. s would be winning as well. Now if it's the Closed Pl.'s turn at s, then $\exists\, a \in A$ s.t. sa is still not winning for the Open Player, o.w. s would be winning for the

Open Player. Thus, following this strategy, the Closed Player creates a play (a_n) s.t. at no finite stage $(a_n)_{n<N}$ does the Open Player have a winning strategy. This implies that $(a_n) \in$ Closed payoff set since o.w., if it's in the Open payoff set, then this would be known at some finite stage $(a_n)_{n<N}$, so Open Player would have the dream winning strategy at that stage, namely "play whatever". $\square$

<u>On Borel determinacy.</u> After Gale-Stewart, it was also proven that F_σ / G_δ sets are determined and the proof was already quite difficult. After a while, in 1975, D. Martin proved the Borel determinacy theorem, i.e. that all Borel sets are determined.

Regularity properties of analytic sets.

<u>Perfect set property.</u> Recall that the determinacy of the

cut-and-choose game implied the PSP for the payoff set. We will use the unravelled version of this game to show the PSP for analytic sets, based on the fact that they are projections of closed sets and closed sets are determined. The PSP property for coanalytic sets is independent of ZFC.

Unravelled cut-and-choose game. Let X be a perfect Polish space and let $P \subseteq X \times \mathbb{N}^{\mathbb{N}}$ be the payoff set. The unravelled cut-and-choose game $G_u(P)$ is:

Player 1.	$(U_0^{(0)}, U_0^{(1)}), y_0$		$(U_1^{(0)}, U_1^{(1)})\ y_1$		$\dots$
Player 2.		i_0		i_1	

Rules: $\overline{U_{n+1}^{(i)}} \subseteq U_n^{i_n}$ and $\operatorname{diam}(U_n^{i}) \to 0$.

Winning: Player 1 wins if $(x, y) \in P$, where $\bigcap_n U_n^{i_n} = \{x\}$ and $y := (y_n)$.

In other words, Pl. 1 plays a witness to x being in $\operatorname{proj}(P)$.

Note that for any Polish space X, taking the perfect kernel, we may assume X is perfect. Any analytic subset $A \subseteq X$ is a projection of a closed subset $F \subseteq X \times \mathbb{N}^{\mathbb{N}}$.

Prop. The map π from the space $M^{\mathbb{N}}$ to $X \times \mathbb{N}^{\mathbb{N}}$, where M is the set of moves of the unravelled cut-and-choose game, mapping each infinite play to its outcome (x, y) is continuous.

Proof. Fix $(x,y) \in$ open set $U \subseteq X \times \mathbb{N}^{\mathbb{N}}$ and find open subset $\hat{U} \subseteq M^{\mathbb{N}}$ s.t. $\pi(\hat{U}) \subseteq U$. Assume that $U = V \times W$, $V \subseteq X$ open and $W \subseteq \mathbb{N}^{\mathbb{N}}$ basic clopen. Observe that at some finite stage of the game, all possible outcomes would be in $V \times W$. $\square$

Theorem. (a) If Pl. I has a winning str. in the unravelled game $G_u(P)$, $P \subseteq X \times \mathbb{N}^{\mathbb{N}}$, then it also has a winning strat. in the usual cut-and-choose game with payoff set $\mathrm{proj}_X(P)$.
Thus, $\mathrm{proj}_X(P)$ contains a homeo. copy of $2^{\mathbb{N}}$.

(b) If Pl. 2 has a winning str. in the $G_u(P)$, then $\mathrm{proj}_X(P)$ is ctbl.

Proof. (a) This is trivial, just don't show the y_n's.

(b) We redo the proof for the unravelled game.

For $(x,y) \in X \times \mathbb{N}^{\mathbb{N}}$, we call an even-length position

$$p = \underbrace{\left(\left(\left(U_0^{(0)}, U_0^{(1)} \right), y_0 \right), i_0, \left(U_1^{(0)}, U_1^{(1)} \right) y_1, i_1, \dots \right)}$$

good for (x,y) if (x,y) is "still a potential outcome of the game after p", i.e. $x \in U_n^{i_n}$ and $y \geq (y_i)_{i<n}$.

Just like before, we observe that $\forall (x,y) \in P$ $\exists$ maximal good position p in the winning str. σ for Player 2.

Thus, $P \subseteq \bigcup_{p \in \sigma} M_p$, where $M_p := \{(x,y) \in P : p \text{ is max-good for } (x,y)\}$. We show that $|\mathrm{proj}_X M_p| \leq 1$ hence otherwise p would have an extension that's still good for one of the pts in M_p. Thus, $\mathrm{proj}_X P \subseteq \bigcup_{p \in \sigma} \mathrm{proj}_X M_p$ is ctbl. $\square$

Cor. Analytic sets have the PSP.

Proof. Let X be WLOG perfect Polish and $A \subseteq X$ analytic, so $A = \mathrm{proj}_X F$, where $F \subseteq X \times \mathbb{N}^{\mathbb{N}}$. Then the unravelled cut-and-choose game $G_u(F)$ is a closed game (by the continuity of the play-to-continuous map), hence determined by Gale-Stewart, hence $\mathrm{proj}_X F$ has the PSP by the previous thm. $\square$